Seymour

Ivan S and Martin Dvorak

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Chapter 1

Code

1.1 TU Matrices

Definition 1 (TU matrix)**.** A rational matrix is *totally unimodular (TU)* if its every subdeterminant (i.e., determinant of every square submatrix) is 0 or ± 1.

Lemma 2 (entries of a TU matrix)**.** *If A is TU, then every entry of A is* 0 *or* ± 1*.*

Proof sketch. Every entry is a square submatrix of size 1, and therefore has determinant (and value) 0 or ± 1. □

Lemma 3 (any submatrix of a TU matrix is TU)**.** *Let A be a rational matrix that is TU and let B be a submatrix of A. Then B is TU.*

Proof sketch. Any square submatrix of B is a submatrix of A, so its determinant is 0 or ± 1. Thus, B is TU. □

Lemma 4 (transpose of TU is TU)**.** *Let A be a TU matrix. Then A^T is TU.*

Proof sketch. A submatrix T of A^T is a transpose of a submatrix of A, so $\det T \in \{0, \pm 1\}$. □

Lemma 5 (appending zero vector to TU)**.** *Let A be a matrix. Let a be a zero row. Then $C = [A/a]$ is TU exactly when A is.*

Proof sketch. Let T be a square submatrix of C, and suppose A is TU. If T contains a zero row, then $\det T = 0$. Otherwise T is a submatrix of A, so $\det T \in \{0, \pm 1\}$. For the other direction, because A is a submatrix of C, we can apply lemma 3. □

Lemma 6 (appending unit vector to TU)**.** *Let A be a matrix. Let a be a unit row. Then $C = [A/a]$ is TU exactly when A is.*

Proof sketch. Let T be a square submatrix of C, and suppose A is TU. If T contains the ± 1 entry of the unit row, then $\det T$ equals the determinant of some submatrix of A times ± 1, so $\det T \in \{0, \pm 1\}$. If T contains some entries of the unit row except the ± 1, then $\det T = 0$. Otherwise T is a submatrix of A, so $\det T \in \{0, \pm 1\}$. For the other direction, simply note that A is a submatrix of C, and use lemma 3. □

Lemma 7 (TUness with adjoint identity matrix)**.** *A is TU iff every basis matrix of $[I \mid A]$ has determinant ± 1.*

Proof sketch. Gaussian elimination. Basis submatrix: its columns form a basis of all columns, its rows form a basis of all rows. □

Lemma 8 (block-diagonal matrix with TU blocks is TU)**.** *Let A be a matrix of the form* $\begin{array}{|c|c|}\hline A_1 & 0 \\ \hline 0 & A_2 \\ \hline \end{array}$ *where A_1 and A_2 are both TU. Then A is also TU.*

Proof sketch. Any square submatrix T of A has the form $\begin{array}{|c|c|}\hline T_1 & 0 \\ \hline 0 & T_2 \\ \hline \end{array}$ where T_1 and T_2 are submatrices of A_1 and A_2, respectively.

- If T_1 is square, then T_2 is also square, and $\det T = \det T_1 \cdot \det T_2 \in \{0, \pm 1\}$.
- If T_1 has more rows than columns, then the rows of T containing T_1 are linearly dependent, so $\det T = 0$.
- Similar if T_1 has more columns than rows.

□

Lemma 9 (appending parallel element to TU)**.** *Let A be a TU matrix. Let a be some row of A. Then $C = [A/a]$ is TU.*

Proof sketch. Let T be a square submatrix of C. If T contains the same row twice, then the rows are $\mathbb{Z}_2$-dependent, so $\det T = 0$. Otherwise T is a submatrix of A, so $\det T \in \{0, \pm 1\}$. □

Lemma 10 (appending rows to TU)**.** *Let A be a TU matrix. Let B be a matrix whose every row is a row of A, a zero row, or a unit row. Then $C = [A/B]$ is TU.*

Proof sketch. Either repeatedly apply Lemmas 5, 6, and 9 or perform a similar case analysis directly. □

Corollary 11 (appending columns to TU)**.** *Let A be a TU matrix. Let B be a matrix whose every column is a column of A, a zero column, or a unit column. Then $C = [A \mid B]$ is TU.*

Proof sketch. C^T is TU by Lemma 10 and construction, so C is TU by Lemma 4. □

Definition 12 ($\mathcal{F}$-pivot)**.** Let A be a matrix over a field $\mathcal{F}$ with row index set X and column index set Y. Let A_{xy} be a nonzero element. The result of a *$\mathcal{F}$-pivot* of A on the *pivot element* A_{xy} is the matrix A' over $\mathcal{F}$ with row index set X' and column index set Y' defined as follows.

- For every $u \in X - x$ and $w \in Y - y$, let $A'_{uw} = A_{uw} + (A_{uy} \cdot A_{xw})/(-A_{xy})$.
- Let $A'_{xy} = -A_{xy}$, $X' = X - x + y$, and $Y' = Y - y + x$.

Lemma 13 (pivoting preserves TUness)**.** *Let A be a TU matrix and let A_{xy} be a nonzero element. Let A' be the matrix obtained by performing a real pivot in A on A_{xy}. Then A' is TU.*

Proof sketch.

- By Lemma 7 A is TU iff every basis matrix of $[I \mid A]$ has determinant ± 1. The same holds for A' and $[I \mid A']$.
- Determinants of the basis matrices are preserved under elementary row operations in $[I \mid A]$ corresponding to the pivot in A, under scaling by ± 1 factors, and under column exchange, all of which together convert $[I \mid A]$ to $[I \mid A']$.

□

Lemma 14 (pivoting preserves TUness)**.** *Let A be a matrix and let A_{xy} be a nonzero element. Let A' be the matrix obtained by performing a real pivot in A on A_{xy}. If A' is TU, then A is TU.*

Proof sketch. Reverse the row operations, scaling, and column exchange in the proof of Lemma 13. □

1.1.1 Minimal Violation Matrices

Definition 15 (minimal violation matrix)**.** Let A be a rational $\{0, \pm 1\}$ matrix that is not TU but all of whose proper submatrices are TU. Then A is called a *minimal violation matrix of total unimodularity (minimal violation matrix).*

Lemma 16 (simple properties of MVMs)**.** *Let A be a minimal violation matrix.*

- *A is square.*
- $\det A \notin \{0, \pm 1\}$.
- *If A is 2×2, then A does not contain a 0.*

Proof sketch.

- If A is not square, then since all its proper submatrices are TU, A is TU, contradiction.
- If $\det A \in \{0, \pm 1\}$, then all subdeterminants of A are 0 or ± 1, so A is TU, contradiction.
- If A is 2×2 and it contains a 0, then $\det A \in \{\pm 1\}$, which contradicts the previous item.

□

Lemma 17 (pivoting in MVMs)**.** *Let A be a minimal violation matrix. Suppose A has ≥ 3 rows. Suppose we perform a real pivot in A, then delete the pivot row and column. Then the resulting matrix A' is also a minimal violation matrix.*

Proof sketch.

- Let A'' denote matrix A after the pivot, but before the pivot row and column are deleted.
- Since A is not TU, Lemma 14 implies that A'' is not TU. Thus A' is not TU by Lemma 8.
- Let T' be a proper square submatrix of A'. Let T'' be the submatrix of A'' consisting of T' plus the pivot row and the pivot column, and let T be the corresponding submatrix of A (defined by the same row and column indices as T'').
- T is TU as a proper submatrix of A. Then Lemma 13 implies that T'' is TU. Thus T' is TU by Lemma 3.

□

1.2 Matroid Definitions

Definition 18 (binary matroid)**.** Let B be a binary matrix, let $A = [I \mid B]$, and let E denote the column index set of A. Let $\mathcal{I}$ be all index subsets $Z \subseteq E$ such that the columns of A indexed by Z are independent over $\mathbb{Z}_2$. Then $M = (E, \mathcal{I})$ is called a *binary matroid* and B is called its *(standard) representation matrix*.

Definition 19 (regular matroid)**.** Let M be a binary matroid generated from a standard representation matrix B. Suppose B has a TU signing, i.e., there exists a rational matrix A such that:

- A is a signed version of B, i.e., $|A| = B$,
- A is totally unimodular.

Then M is called a *regular matroid*.

Lemma 20 (regularity is ignostic of representation)**.**

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1.3 k-Separation and k-Connectivity

Definition 21 (k-separation)**.** Let M be a binary matroid generated by a standard representation matrix B. Suppose that B is partitioned as

	Y_1	Y_2
X_1	B_1	D_2
X_2	D_1	B_2

where $X_1 \sqcup X_2$ is a partition of the rows of B and $Y_1 \sqcup Y_2$ is a partition of its columns. Let $k \in \mathbb{Z}_{\geq 1}$ and suppose that

- $|X_1 \cup Y_1| \geq k$ and $|X_2 \cup Y_2| \geq k$,
- $\mathbb{Z}_2$-rank $D_1 + \mathbb{Z}_2$-rank $D_2 \leq k - 1$.

Then $(X_1 \cup Y_1, X_2 \cup Y_2)$ is called a *(Tutte) k-separation* of B and M.

Definition 22 (exact k-separation)**.** A k-separation is called *exact* if the rank condition holds with equality.

Definition 23 (k-separability)**.** We say that B and M are *(exactly) (Tutte) k-separable* if they have an (exact) k-separation.

Definition 24 (k-connectivity)**.** For $k \geq 2$, M and B are *(Tutte) k-connected* if they have no ℓ-separation for $1 \leq \ell < k$. When M and B are 2-connected, they are also called *connected*.

1.4 Sums

1.4.1 1-Sums

Definition 25 (1-sum of matrices)**.** Let B be a matrix that can be represented as

	Y_1	Y_2
X_1	B_1	0
X_2	0	B_2

Then we say that B_1 and B_2 are the two *components* of a 1-*sum decomposition* of B.

Conversely, a 1-*sum composition* with *components* B_1 and B_2 is the matrix B above.

The expression $B = B_1 \oplus_1 B_2$ means either process.

Definition 26 (matroid 1-sum)**.** Let M be a binary matroid with a representation matrix B. Suppose that B can be partitioned as in Definition 25 with non-zero blocks B_1 and B_2. Then the binary matroids M_1 and M_2 represented by B_1 and B_2, respectively, are the two *components* of a 1-*sum decomposition* of M.

Conversely, a 1-*sum composition* with *components* M_1 and M_2 is the matroid M defined by the corresponding representation matrix B.

The expression $M = M_1 \oplus_1 M_2$ means either process.

Lemma 27 (1-sum is commutative)**.**

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Theorem 28 (1-sum of regular matroids is regular)**.** *Let M_1 and M_2 be regular matroids. Then $M = M_1 \oplus_1 M_2$ is a regular matroid.*

Conversely, if a regular matroid M can be decomposed as a 1-sum $M = M_1 \oplus_1 M_2$, then M_1 and M_2 are both regular.

Proof sketch.

extract into lemmas about TU matrices

Let B, B_1, and B_2 be the representation matrices of M, M_1, and M_2, respectively.

- Converse direction. Let B' be a TU signing of B. Let B_1' and B_2' be signings of B_1 and B_2, respectively, obtained from B. By Lemma 3, B_1' and B_2' are both TU, so M_1 and M_2 are both regular.

- Forward direction. Let B_1' and B_2' be TU signings of B_1 and B_2, respectively. Let B' be the corresponding signing of B. By Lemma 8, B' is TU, so M is regular.

□

Lemma 29 (left summand of regular 1-sum is regular)**.**

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Proof. □

Lemma 30 (right summand of regular 1-sum is regular)**.**

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Proof. □

1.4.2 2-Sums

Definition 31 (2-sum of matrices)**.** Let B be a matrix of the form $\begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A_1 & 0 \\ \hline X_2 & D & A_2 \\ \hline \end{array}$ Let B_1 be a matrix of the form $\begin{array}{c|c|} & Y_1 \\ X_1 & A_1 \\ \text{Unit} & x \end{array}$ Let B_2 be a matrix of the form $\begin{array}{c|c|c|} & \text{Unit} & Y_2 \\ \hline X_2 & y & A_2 \\ \hline \end{array}$ Suppose that $\mathbb{Z}_2$-rank $D = 1$, $x \neq 0$, $y \neq 0$, $D = y \cdot x$ (outer product).

Then we say that B_1 and B_2 are the two *components* of a 2-*sum decomposition* of B.

Conversely, a 2-*sum composition* with *components* B_1 and B_2 is the matrix B above.

The expression $B = B_1 \oplus_2 B_2$ means either process.

Definition 32 (matroid 2-sum)**.** Let M be a binary matroid with a representation matrix B. Suppose B, B_1, and B_2 satisfy the assumptions of Definition 31. Then the binary matroids M_1 and M_2 represented by B_1 and B_2, respectively, are the two *components* of a 2*-sum decomposition* of M.

Conversely, a 2*-sum composition* with *components* M_1 and M_2 is the matroid M defined by the corresponding representation matrix B.

The expression $M = M_1 \oplus_2 M_2$ means either process.

Lemma 33 (2-sum of TU matrices is a TU matrix)**.** *Let B_1 and B_2 be TU matrices. Then $B = B_1 \oplus_2 B_2$ is a TU matrix.*

Proof sketch.

Let B_1' and B_2' be TU signings of B_1 and B_2, respectively. In particular, let A_1', x', A_2', and y' be the signed versions of A_1, x, A_2, and y, respectively. Let B' be the signing of B where the blocks of A_1 and A_2 are signed as A_1' and A_2', respectively, and the block of D is signed as $D' = y' \cdot x'$ (outer product).

Note that $[A_1'/D']$ is TU by Lemma 10, as every row of D' is either zero or a copy of x'. Similarly, $[D' \mid A_2']$ is TU by Corollary 11, as every column of D' is either zero or a copy of y'. Additionally, $[A_1' \mid 0]$ is TU by Corollary 11, and $[0/A_2']$ is TU by Lemma 10.

prove lemma below, separate into statement about TU matrices

Lemma: Let T be a square submatrix of B'. Then $\det T \in \{0, \pm 1\}$.

Proof: Induction on the size of T. *Base:* If T consists of only 1 element, then this element is 0 or ± 1, so $\det T \in \{0, \pm 1\}$. *Step:* Let T have size t and suppose all square submatrices of B' of size $\leq t - 1$ are TU.

- Suppose T contains no rows of X_1. Then T is a submatrix of $[D' \mid A_2']$, so $\det T \in \{0, \pm 1\}$.
- Suppose T contains no rows of X_2. Then T is a submatrix of $[A_1' \mid 0]$, so $\det T \in \{0, \pm 1\}$.
- Suppose T contains no columns of Y_1. Then T is a submatrix of $[0/A_2']$, so $\det T \in \{0, \pm 1\}$.
- Suppose T contains no columns of Y_2. Then T is a submatrix of $[A_1'/D']$, so $\det T \in \{0, \pm 1\}$.
- Remaining case: T contains rows of X_1 and X_2 and columns of Y_1 and Y_2.
- If T is 2×2, then T is TU. Indeed, all proper submatrices of T are of size ≤ 1, which are $\{0, \pm 1\}$ entries of A', and T contains a zero entry (in the row of X_2 and column of Y_2), so it cannot be a minimal violation matrix by Lemma 16. Thus, assume T has size ≥ 3.
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  complete proof, see last paragraph of Lemma 11.2.1 in Truemper

□

Theorem 34 (2-sum of regular matroids is a regular matroid)**.** *Let M_1 and M_2 be regular matroids. Then $M = M_1 \oplus_2 M_2$ is a regular matroid.*

Proof sketch. Let B, B_1, and B_2 be the representation matrices of M, M_1, and M_2, respectively. Apply Lemma 33. □

Lemma 35 (left summand of regular 2-sum is regular)**.**

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Lemma 36 (right summand of regular 2-sum is regular)**.**

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1.4.3 3-Sums

Definition 37 (3-sum of matrices)**.**

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Definition 38 (matroid 3-sum)**.**

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Theorem 39 (3-sum of regular matroids is regular)**.**

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Lemma 40 (left summand of regular 3-sum is regular)**.**

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Lemma 41 (right summand of regular 3-sum is regular)**.**

add